Problem Section 9

Edited by Manjil P. Saikia

School of Mathematics, Cardiff University, CF24 4AG, UK

E-mail: manjil@saikia.in

This section contains unsolved problems, whose solutions we ask from the readers, which we will publish in the subsequent issues. All solutions should preferably be typed in LaTeX and emailed to the editor. If you would like to propose problems for this section then please send your problems (with solutions) to the above mentioned email address, preferably typed in LaTeX. Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by *30 March, 2023*. If a problem is not original, the proposer should inform the editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in Ganit Bikash. Solvers are asked to include references for any non-trivial results they use in their solutions.

Problem 18. *Proposed by Manjil P. Saikia (Cardiff University)*

A natural number n is called a k_p-perfect number if the following holds

$$\sigma(n) = \left(\frac{p}{p-1}\right) n,$$

where $\sigma(n)$ is the sum of divisors function. Prove that there are no k_p-perfect numbers for $p \geq 5$. Further, show that the only k_3-perfect number is 2.

Solutions to Old Problems

We received correct solutions to Problem 15 from **Bishwajit Sarma** (University of Hyderabad), **Amit Kumar Basistha** (Indian Statistical Institute, Bengaluru) and **Dr. Kuldeep Sarma** (Tezpur University). We received no solutions for Problem 16.

Solution 15. *The solution below is by* ***Bishwajit Sarma*** *(University of Hyderabad) and* ***Dr. Kuldeep Sarma*** *(Tezpur University).*

Note that

$$(n!)^2 = [1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n][1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n].$$

By grouping terms in pairs, we have

$$(n!)^2 = \prod_{i=1}^{n} i(n+1-i).$$

Now observe that $i(n+1-i) \geq n$ for every $i \in \{1, 2, ..., n\}$. So we have

$$(n!)^2 = \prod_{i=1}^{n} i(n+1-i) \geq n^n. \tag{0.1}$$

Now let us apply AM-GM inequality to the numbers $1, 2, \dots, n$ to get

$$\sqrt[n]{n}! = \sqrt[n]{1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n} \leq \frac{1+2+\ldots+(n-1)+n}{n} = \frac{n+1}{2}. \tag{0.2}$$

From equations (0.1) and (0.2), we have

$$n^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^n}.$$

Solution 16. *Solution by the proposer, Dr. Manjil P. Saikia.*

Take $n = 2m$ with $m \in \mathbb{Z}_{\geq 7}$. We have to show

$$\sum_{k=1}^{m-3} \left\lfloor \frac{m-k-1}{2} \right\rfloor > 1 + \sum_{k=1}^{\lfloor \frac{m-1}{3} \rfloor} \left\lfloor \frac{m-3k+1}{2} \right\rfloor + \sum_{k=1}^{\lfloor \frac{m-3}{3} \rfloor} \left\lfloor \frac{m-3k-1}{2} \right\rfloor. \tag{0.3}$$

Note that

$$\begin{aligned}\sum_{k=1}^{m-3} \left\lfloor \frac{m-k-1}{2} \right\rfloor = \sum_{k=0}^{m-4} \left\lfloor \frac{k+2}{2} \right\rfloor &> \sum_{k=0}^{m-4} \left(\frac{k+2}{2} - 1 \right) \left(\text{since, } \lfloor x \rfloor > x-1 \right) \\ &= \frac{(m-4)(m-3)}{4} = \frac{m^2-7m+12}{4},\end{aligned}$$

> "I have seen numerous instances of young researchers unwilling to work on a problem until they feel that they have mastered all the theory that is associated with it. I don't work this way."
>
> **– Richard P. Stanley**

and

$$
\begin{aligned}
&1+\sum_{k=1}^{\lfloor\frac{m-1}{3}\rfloor}\left\lfloor\frac{m-3k+1}{2}\right\rfloor+\sum_{k=1}^{\lfloor\frac{m-3}{3}\rfloor}\left\lfloor\frac{m-3k-1}{2}\right\rfloor\\
&<1+\sum_{k=1}^{\lfloor\frac{m-1}{3}\rfloor}\frac{m-3k+1}{2}+\sum_{k=1}^{\lfloor\frac{m-3}{3}\rfloor}\frac{m-3k-1}{2}\ \left(\text{since, } \lfloor x\rfloor<x\right)\\
&=1+\frac{m+1}{2}\left\lfloor\frac{m-1}{3}\right\rfloor-\frac{3}{4}\left\lfloor\frac{m-1}{3}\right\rfloor\left(\left\lfloor\frac{m-1}{3}\right\rfloor+1\right)+\frac{m-1}{2}\left\lfloor\frac{m-3}{3}\right\rfloor\\
&\qquad-\frac{3}{4}\left\lfloor\frac{m-3}{3}\right\rfloor\left(\left\lfloor\frac{m-3}{3}\right\rfloor+1\right)\\
&<1+\frac{m+1}{2}\frac{m-1}{3}-\frac{3}{4}\left(\frac{m-1}{3}-1\right)\left(\frac{m-1}{3}\right)+\frac{m-1}{2}\frac{m-3}{3}-\frac{3}{4}\left(\frac{m-3}{3}-1\right)\left(\frac{m-3}{3}\right)\\
&=\frac{m^2+3m-3}{6}.
\end{aligned}
$$

Now, $\dfrac{m^2-7m+12}{4}>\dfrac{m^2+3m-3}{6}$ for $m\in\mathbb{Z}_{\geq 26}$. We finish the proof by checking the inequality for $7\leq m\leq 25$ numerically in Mathematica.

The proof of the next inequality is similar to the above and we leave it to the reader. Both these inequalities appear in a recent paper as an auxiliary lemma to solve a combinatorial problem. For more details see [1, Lemmas 2.1 and 2.2] or [2, Lemma 2.1].

[1] Koustav Banerjee, Sreerupa Bhattacharjee, Manosij Ghosh Dastidar, Pankaj Jyoti Mahanta, and Manjil P. Saikia. Parity biases in partitions and restricted partitions. *European J. Combin.*, 103:Paper No. 103522, 19, 2022.

[2] Koustav Banerjee, Sreerupa Bhattacharjee, Manosij Ghosh Dastidar, Pankaj Jyoti Mahanta, and Manjil P. Saikia. Parity biases in partitions and restricted partitions. *Sém. Lothar. Combin.*, 86B:Art. 21, 12, 2022.

"If six people come to a party, then either there are three who know each other or three who do not. Five are not enough. I heard this problem when I was about 14 or 15; and I have never been the same person again."

– Maria Chudnovsky